Viva For Mechanics

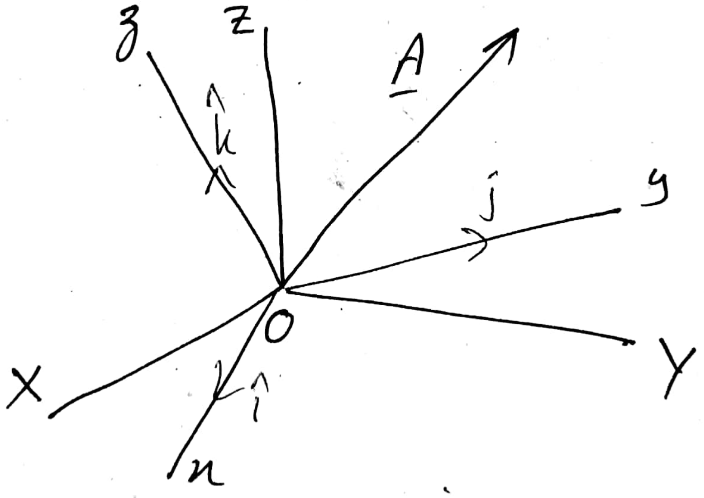
Inertial frame of refence.

frame of refence in which newton's 1st law of motion is valid.

Non-inertial Frame of reference.

In which newton's 1st law is not valid. It is accelerated and inertial or fictitious forces arise

Rotating Axes Theorem



Let XYZ be an inertial Co-ordinate System with origin O. Suppose co-ordinate System xyz having the same origin be rotating wrt. XYZ System,

If $\underline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ a vector

changing with time.

To an observer fixed relative to xyz system (i.e in xyz-system), time derivative of $\underline{A}$ is

$$\left(\frac{d\underline{A}}{dt}\right)_r = \frac{dA_1}{dt}\hat{i} + \frac{dA_2}{dt}\hat{j} + \frac{dA_3}{dt}\hat{k}$$

Time derivative of $\underline{A}$ relative to XYZ system symbolised by $\left(\frac{d\underline{A}}{dt}\right)_f$ is

$$\left(\frac{d\underline{A}}{dt}\right)_f = \left(\frac{d\underline{A}}{dt}\right)_r + \underline{\omega} \times \underline{A}$$

$\underline{\omega}$ is angular velocity of xyz system with respt to XYZ system.

If $D_f = \left(\frac{d}{dt}\right)_f$ $\quad D_r = \left(\frac{d}{dt}\right)_r$

, then $D_f = D_r + \underline{\omega} \times$

If $\underline{A}$ is the posn vector $\underline{r}$ of particle, then

$$\left(\frac{d\underline{r}}{dt}\right)_f = \left(\frac{d\underline{r}}{dt}\right)_r + \underline{\omega} \times \underline{r}$$

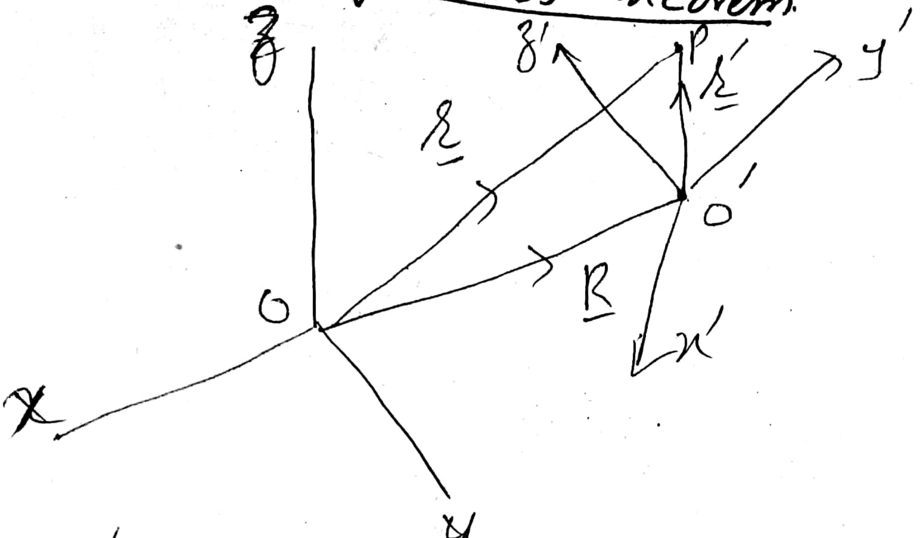
$$v_{p/f} = v_{p/r} + v_{r/f}$$

$v_{p/f}$ = velocity of particle relative to fixed syst.

$v_{p/r}$ = velocity of particle - - moving syst.

$v_{r/f}$ = velocity of moving system relative to fixed sys

Moving Axes theorem



x'y'z' is translate as well rotating.
If $\underline{R}$ is p.v. of orig O' of x'y'z' system at any time t, then p.v $\underline{r}$ of particle P w.r.t. xyz system is

$$\underline{r} = \underline{r}' + \underline{R}$$

$$\left(\frac{d\underline{r}}{dt}\right)_f = \left(\frac{d\underline{r}'}{dt}\right)_f + \left(\frac{d\underline{R}}{dt}\right)_f$$

$$= \left(\frac{d\underline{r}'}{dt}\right)_r + \underline{\omega} \times \underline{r}' + \dot{\underline{R}}$$

$$\underline{v} = \dot{\underline{R}} + \underline{v}' + \underline{\omega} \times \underline{r}'$$

Note (1) in rotating axes there is

$$\left(\frac{d\underline{r}}{dt}\right)_f = \left(\frac{d\underline{r}}{dt}\right)_r + \underline{\omega} \times \underline{r}$$

$$\left(\frac{d}{dt}\right)_f = \left(\frac{d}{dt}\right)_r + \underline{\omega} \times$$

Acc of particle is given wrt fixed axes as

$$\underline{a} = \left(\frac{d^2\underline{r}}{dt^2}\right)_f = \left(\frac{d}{dt}\right)_f \left(\frac{d\underline{r}}{dt}\right)_f$$

$$= \left(\frac{d}{dt}\right)_f \left[\left(\frac{d\underline{r}}{dt}\right)_r + \underline{\omega} \times \underline{r} \right]$$

$$= \left[\left(\frac{d}{dt}\right)_r + \underline{\omega} \times \right] \left[\left(\frac{d\underline{r}}{dt}\right)_r + \underline{\omega} \times \underline{r} \right]$$

$$= \left(\frac{d}{dt}\right)_r \left[\left(\frac{d\underline{r}}{dt}\right)_r + \underline{\omega} \times \underline{r} \right] + \underline{\omega} \times \left[\left(\frac{d\underline{r}}{dt}\right)_r + \underline{\omega} \times \underline{r} \right]$$

$$= \left(\frac{d^2\underline{r}}{dt^2}\right)_r + \left(\frac{d(\underline{\omega} \times \underline{r})}{dt}\right)_r + \underline{\omega} \times \left(\frac{d\underline{r}}{dt}\right)_r + \underline{\omega} \times \underline{\omega} \times \underline{r}$$

$$= \left(\frac{d^2\underline{r}}{dt}\right)_r + \underline{\omega} \times \left(\frac{d\underline{r}}{dt}\right)_r + \frac{d\underline{\omega}}{dt} \times \underline{r} + \underline{\omega} \times \left(\frac{d\underline{r}}{dt}\right)_r$$

$$+ \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

$$= \left(\frac{d^2\underline{r}}{dt}\right)_r + \frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \left(\frac{d\underline{r}}{dt}\right)_r$$

$$+ \underline{\omega} \times \underline{\omega} \times \underline{r}$$

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System of particles

Discrete System of particles

System of particles is discrete if particles can be considered as seperated frome each other

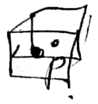
Continuous System of particles

System of particles is continuous if particles can not be considered as seperated from each other

Density

(1) **Volume Density** Mass per unit volume is called volume density.
Mathematically If ΔM total mass of volume ΔV, then density ρ is

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V}$$



The density is a function of position & can very from point to point. when density is constant system is said to be uniform

(2) **Surface Density** when continuous system of particles occupy a surface, we define surface density or mass per unit area.

$$\rho = \lim_{\Delta A \to 0} \frac{\Delta M}{\Delta A}$$

(3) Linear Density: when mass is distributed on line or [curve] we define linear density or mass per unit length

$$\rho = \lim_{\Delta l \to 0} \frac{\Delta M}{\Delta l}$$

Elastic Bodies

If the force applied to system of particles change the distance bet. individual particles, then such system are called deformable or elastic bodies

Rigid Body

A system of particles in which the distance bet. any two particles does not change regardless of the forces acting ~~and~~ is called a rigid body

Degrees of Freedom

The no of co-ordinates required to specify the position of a system of one or more particles is called degree of freedom of the system. e.g A particle moving freely in space requires 3 co ordinates (x, y, z). So no of degree of freedom = 3

Mass elements

(1) In case of thin uniform rod, of density ρ, we have

$$dm = \rho\, dx$$

where dx element of differential length.

(2) In case of flat lamina.

$$dm = \rho\, dx\, dy = \rho\, dA$$

where $dA = dx\, dy$, ρ is density at point (x, y).

for polar coordinates

$$dm = \rho\, r\, d\theta\, dr$$

(3) In case of solid.

$$dm = \rho\, dx\, dy\, dz = \rho\, dV$$

for spherical polar co-ordinates

$$dm = \rho \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Centre of Mass

The centre of mass or centroid of system of particles is a point C having position vector

$$\underline{r}_c = \frac{\sum_{i=1}^{N} m_i \underline{r}_i}{\sum m_i} = \frac{\sum m_i \underline{r}_i}{M}$$

where $\underline{r}_i$ are p.v of N-particles

& $\sum_{i=1}^{N} m_i \underline{r}_i$ = sum of mass moments

Note: Sum of mass moments relative to c.m. is zero i.e.

$$\sum_{i=1}^{n} m_i \underline{r}_i' = 0$$ where $\underline{r}_i'$ are p.v. relative to c.m. c.

For continuous distribution of mass over region R of space

$$\underline{r}_c = \frac{\int_R \underline{r}\, dm}{\int_R dm}$$

$$= \frac{\int_R \rho \underline{r}\, dV}{\int_R \rho\, dV}$$

If $\underline{r}_c = \bar{x}\hat{i} + \bar{y}\hat{j} + \bar{z}\hat{k}$, then

$$\bar{x} = \frac{\sum m_i x_i}{M} \qquad \bar{y} = \frac{\sum m_i y_i}{M}$$

$$\bar{z} = \frac{\sum m_i z_i}{M}$$

$$\bar{x} = \frac{\int_R \rho x\, dV}{M} \qquad \bar{y} = \frac{\int \rho y\, dV}{M}$$

$$\bar{z} = \frac{\int_R \rho z\, dV}{M}.$$

Centre of Gravity

If a system of particles is in a uniform gravitational field, the centre of mass is sometimes called centre of gravity.

Momentum of a System of particles

$$\underline{v}_i = \frac{d\underline{r}_i}{dt}$$ is the velocity of m_i

then total momentum is

$$p = \sum_{i=1}^{n} m_i v_i = \sum_{i=1}^{n} m_i \dot{\underline{r}}_i$$

$$= \sum m_i \frac{d}{dt} \underline{r}_i$$

$$= \frac{d}{dt} \left(\sum m_i \underline{r}_i \right)$$

$$= \frac{d}{dt} \left(M \underline{r}_c \right)$$

$$= M \frac{d\underline{r}_c}{dt} = M \dot{\underline{r}}_c = M \underline{V}_c$$

where $\underline{v}_c = \frac{d\underline{r}_c}{dt}$ is velocity of c.m

Thus total momentum of a system can be found by multiplying total mass M by the velocity $\underline{v}_c$ of c.m.

Motion of centre of Mass

If the internal forces b/w any two particles of a system obey 3rd law & cancel out, then the resultant external force F on system

$$\underline{F}_i = m_i \underline{a}_i$$

$$= m_i \frac{d\underline{v}_i}{dt} =$$

$$= \frac{d}{dt}(m_i v_i) = \frac{d^2}{dt^2}\{\sum m_i r_i\}$$

$$\underline{F}_i = \frac{d}{dt} P_i$$

Summing over all particles

$$F = \sum F_i = \frac{d^2}{dt^2}\{\sum m_i r_i\}$$

$$= \frac{d^2}{dt^2}\{M \underline{r}_c\}$$

$$\underline{F} = M\frac{d^2 r_c}{dt^2} = M\frac{dV_c}{dt}$$

⇒ The c-m of the system of particles moves as if the total Mass and resultant external force were applied at that point

<u>Conservation of Momentum</u>

$$\underline{F} = \frac{d}{dt}(\sum m_i \underline{v}_i)$$

when total external force $\underline{F} = 0$

, then

$$\frac{d}{dt}(\sum m_i \underline{v}_i) = 0$$

$$\Rightarrow \underline{P} = \sum m_i \underline{v}_i = \text{Constant}$$

⇒ If total external force a system is zero, then total momentum remains constant. In such case c-m is either at rest or moves with uniform velocity.

Angular Momentum

for i'th particle

$$\underline{L}_i = \underline{r}_i \times \underline{p}_i$$
$$= \underline{r}_i \times m_i \underline{v}_i$$
$$= \text{moment of momentum.}$$

= moment of system.

total angular momentum of system

$$\underline{L} = \sum \underline{r}_i \times m_i \underline{v}_i$$
$$= \sum m_i (\underline{r}_i \times \underline{v}_i)$$

Total External torque

If $\underline{F}_i$ is the external force acting on i'th particle, then $\underline{r}_i \times \underline{F}_i$ is called moment or torque of the force $\underline{F}_i$ about O. The

$$\underline{T} = \underline{N} = \sum_{i=1}^{n} \underline{r}_i \times F_i$$

is called total external torque about origin

Relation b/w total torque & Angular momentum

If the internal forces b/w any two particles are always directed along the line joining particles [they are central forces] then

$$\underline{N} = \frac{dL}{dt}$$

Conservation of Angular Momentum

$$\underline{N} = \frac{d\underline{L}}{dt}$$

If $\underline{N} = 0$, then $\underline{L}$ = Constant

If the resultant external torque on system of particles is zero, the total angular momentum of the system is constant. It is called principle of conservation of system.

K.E of a system of particles

$$T = \frac{1}{2}\sum m_i v_i^2 = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{r}_i^2$$

Motion Relative to C.m

(1) Total linear momentum of a system of particles about c.m is zero.

$$\sum m_i \underline{v}_i' = \sum m_i \dot{\underline{r}}_i' = 0$$

(2)

$$\underline{L}_0 = \underline{r}_c \times M\underline{V}_c + \sum_{i=1}^{n} m_i(\underline{r}_i' \times \underline{v}')$$

$$= \underline{r}_c \times M\underline{V}_c + \underline{L}_c$$

Angular momentum about O = angular momentum of total mass assumed to be at c.m. plus angular momentum about c.m.

$$T = \frac{1}{2} M V_c^2 + \frac{1}{2} \sum m_i v_i'^2$$

total K.E about any point O
= K.E of translation of c.m. [assuming total mass located there)
+ K.E of motion about c.m.

(4) $$\underline{N}' = \frac{d\underline{L}'}{dt}$$

total external torque about c.m is equal to the time rate of change of angular momentum about c.m.

If motion is described relative to points other than c.m. the above results are more complicated.

Plane Motion of Rigid body

Translation and rotations

A displacement of rigid body in which all points of a rigid body move in parallel lines parallel to each the displacement is called a translation.

If points move in congruent curves then it is called curvilinear translation.

Note in each case of translation motion of body is completely specified by motion of any particle

If during displacement all points of body on some line remain fixed, the displacement is called rotation about the line. All particles move circular paths about axis of rotation and all lines (including those that do not pass through axis) rotate through the same angles in the same time.

Note Rotations can be considered as finite or infinitesimal. finite rotation can not be represented by vectors since commutative law fails. Infinitesimal rotations can be represented by vectors

Euler's Theorem (instantaneous axis of rotation)

A rotation of a rigid body about a fixed point point of body is equivalent to a rotation about a line passing through the point

General Motion of rigid body (Chasles's theorem)

In general motion of rigid body no point of the body may be fixed. In such case Chasles's theorem is fundamental

Theorem The general motion of rigid body can be considered as a translation plus a rotation about a suitable point which is often taken to be c.m

Plane Motion of Rigid Body

The motion of a rigid body is simplified when all points move parallel to a given fixed plane.

In such case two types motion called plane motion are possible.

(1) <u>Rotation about a fixed axis</u>

Rigid body rotation about a fixed axis ⊥ar to fixed plane. System has only one degree of freedom and only one coordinates is required to describe motion.

(2) <u>General plane Motion.</u>

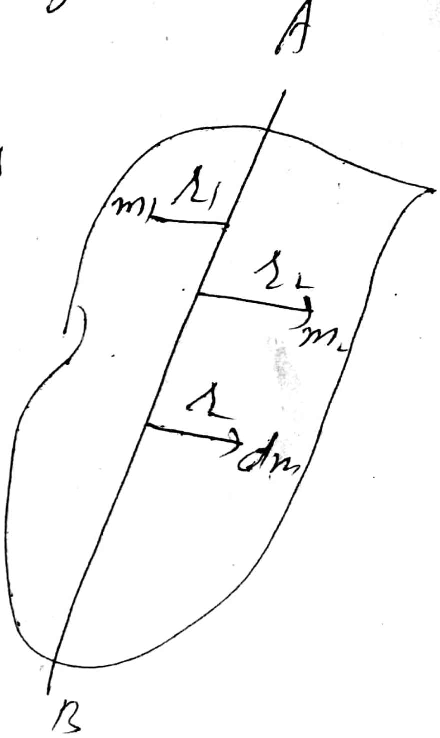
Such motion can be considered as a translation ∥al to a given fixed plane plus a rotation about a suitable axis ⊥ar to the plane. This axis is often so chosen that it passes through c.m. The no degrees of freedom is 3 two for translation & one for rotation.

The axis is referred as instantaneous axis and point where instantaneous axis intersects the fixed plane is called instantaneous centre of rotation.

<u>Note</u> The motion of rigid body in three dimension is more complicated.

Moment of Inertia

Geometric quantity which is impor[tant] in discussing the motion of rigid body is called moment of inertia



The moment of inertia of a mass m about a line or axis AB is defined as

$$I = mr^2$$

where r is distance of mass from line.

The m.I of a system of particles of masses $m_1, m_2 \ldots m_n$ about axis AB is

$$I = \sum_{i=1}^{n} m_i r_i^2$$

where $r_1, r_2 \ldots r_n$ are respective distances from the line.

The m.I of continuous distribution of mass such a solid body R is given by

$$I = \int_R r^2 \, dm$$

where r is distance of element of mass dm from AB

Radius of Gyration

It is customary to express the m.I of inertia about an axis in terms of a characteristic distance k, called radius of gyration of that body about that axis

$$I = \sum m_i r_i^2 = \sum m_i \times k^2 = Mk^2$$

where M = total mass of the body. The radius of Gyration $k = \sqrt{\frac{I}{M}}$ this can be defined as

That distance of a point from the axis of rotation at which if the whole mass (M) is supposed to be concentrated, its m.I about the given axis would be same as with its actual distribution of mass or it represents the distance from the axis at which all the ~~could~~ mass could be concentrated without changing the m.I

Note (1) the m.I of body about an axis depends on the choice of origin of the Reference System

(2) m.I depends upon distribution of mass and its distance from axis of rotation.

M.I of System about Axes

(1) $I_{xx} = \sum_i m_i (y_i^2 + z_i^2)$

(2) $I_{yy} = \sum_i m_i (x_i^2 + z_i^2)$

(3) $I_{zz} = \sum_i m_i (x_i^2 + y_i^2)$

For Continuous Distribution of mass

(1) $I_{xx} = \int_R (y^2 + z^2)\, dm$

(2) $I_{yy} = \int_R (x^2 + z^2)\, dm$

(3) $I_{zz} = \int_R (x^2 + y^2)\, dm$

Polar Moment of inertia

The sum

$I_O = I_x + I_y$

is called the polar moment of inertia or the moment of inertia w.r.t origin

Parallel axes Theorem

The M.I of a rigid body about a given axis is equal to the sum of the moment of inertia of body about a ||el axis passing through C.m and the M.I of the centre of mass body about given axis of rotation assuming the whole mass of body to be concentrated at C.m

Perpendicular Axes Theorem
or
Lamina Theorem

M.I of a plane lamina about an axis ⊥or to the plane of lamina is equal to the sum of moments of inertia of lamina about two axes ⊥or to each other in its own plane and intersecting each other at point where ⊥or axis passes through it

Routh's Rule

M.I about an axis of Symmetry

$$= \text{Mass}\left[\frac{\text{The sum of square of } \perp\text{or semi-axes}}{3 \text{ or } 4 \text{ or } 5}\right]$$

symmetry

⇓ K^2

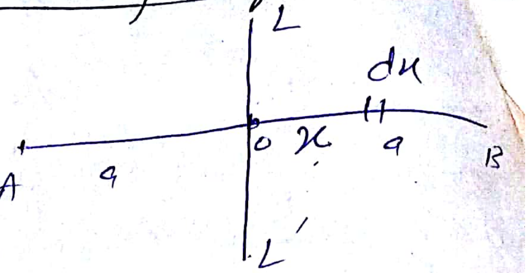
(1) The denominator is 3 if body is rectangular (including rod & square)

(2) The Denominator is 4 if body is elliptical (including a circle)

(3) The Denominator is 5 if body is ellipsoidal (including sphere)

M.I of uniform Rod of length 2a

(I) About an axis which bisects the rod ⊥arly at O



$$\rho = \frac{M}{2a}$$

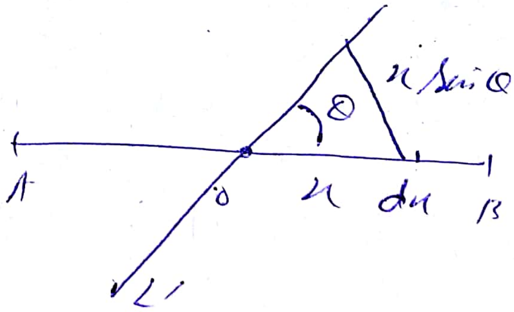
$$dm = \rho dx = \frac{M}{2a} dx$$

$$dI_{LL'} = x^2 dm$$

$$I_{LL} = \int_{-a}^{a} x^2 dm = \frac{M}{2a}\left[\frac{x^3}{3}\right]_{-a}^{a}$$

$$I_{LL} = \boxed{\frac{1}{3}Ma^2}$$

(II) About an axis through mid pt and inclined at angle θ.



$$dm = \rho dx$$

$$dI_{LL'} = x^2 \sin^2\theta \, dm$$

$$I_{LL'} = \int_{-a}^{a} x^2 \sin^2\theta \, \rho . dx$$

$$= \frac{M}{2a} \sin^2\theta \left[\frac{x^3}{3}\right]_{-a}^{a} = \boxed{\frac{1}{3}Ma^2 \sin^2\theta}$$

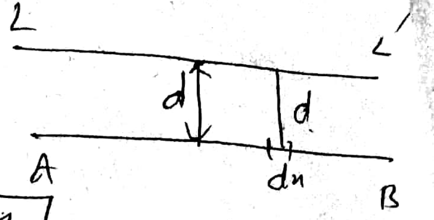
(III) About an axis end pt A or B ⊥arly

$$I_{LL'} = \int_0^{2a} x^2 dm = \frac{4}{3}Ma^2$$

OR By parallel axis theorem.

$$I_{LL'} = \frac{1}{3}Ma^2 + Ma^2 = \frac{4}{3}Ma^2$$

(IV) About an axis parallel to rod at distance (d)



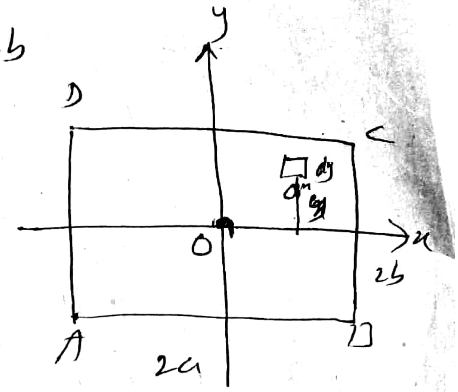
$dI_{LL'} = d^2\, dm$

$I_{LL'} = \int d^2\, dm$

$= d^2 \int dm = \boxed{d^2 M}$

Uniform Rectangular Lamina
Sides 2a & 2b

(I) x - y axes are centroidal axes ||al to sides
Consider area element dx dy at pt (x, y).



$dm = \rho\, dA = \rho\, dx\, dy$

$\rho = \frac{M}{\text{area}} = \frac{M}{4ab}$

$dI_{xx} = y^2\, dm = y^2 \rho . dx\, dy =$

$I_{xx} = \rho \int_{-b}^{b} \int_{-a}^{a} y^2\, dx\, dy$

$= \rho \left[\frac{y^3}{3}\right]_{-b}^{b} \int_{-a}^{a} dx$

$= \rho \left[\frac{b^3}{3} + \frac{b^3}{3}\right] [x]_{-a}^{a} = \rho . \frac{2b^3}{3} . 2a$

$= \frac{4ab^3}{3} . \frac{M}{4ab} = \boxed{\frac{1}{3} Mb^2}$

$I_{yy} = \boxed{\frac{1}{3} Ma^2}$

(2) About an axis passing through centre and perp to lamina

By $\perp$ar axes theorem

$$I_{zz} = I_x + I_y = \frac{1}{3}Ma^2 + \frac{1}{3}Mb^2$$

$$I_{zz} = \boxed{\frac{1}{3}M(a^2+b^2)}$$

(3) M.I about side AB

$$I_{AB} = I_{yy} + Mb^2 = \frac{1}{3}Mb^2 + Mb^2$$

$$= \frac{4}{3}Mb^2$$

$$I_{AD} = I_{xx} + Ma^2 = \frac{1}{3}Ma^2 + Ma^2$$

$$= \frac{4}{3}Ma^2$$

(4) Product of inertia wrt centroial axes

$$I_{xy} = \int_{-b}^{b}\int_{-a}^{a} xy\, dm$$

$$= \int_{-b}^{b}\int_{-a}^{a} xy\, \rho\, dx\, dy$$

$$= 0 \qquad \text{odd function of } x \,\&\, y$$

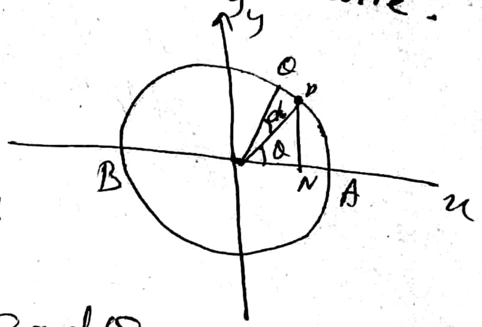
perp

M.I of uniform Ring or Hoop OR uniform circular wire.

radius = a

$\angle xOP = \theta$

$PN = a\sin\theta$



$\rho = \frac{M}{\text{length}} = \frac{M}{2\pi a}$

$\widehat{PQ} = ds = a\,d\theta$

$dm = \rho\, ds = \rho a\, d\theta$

$dI_{AB} = a^2\sin^2\theta\, dm$

$= a^2\sin^2\theta \cdot \rho a\, d\theta$

$= \rho a^3 \sin^2\theta\, d\theta$

$I_{AB} = I_{xx} = a^3\rho \int_0^{2\pi} \sin^2\theta\, d\theta$

$$\boxed{I_{AB} = \frac{1}{2}Ma^2}$$

By Dr - Routh's Rule.

$I_{AB} = M$(Sum of squares of semi-axes $\perp$ to AB)

$= M(a^2)$

$$\boxed{I_{\text{diameter}} = \frac{1}{2}Ma^2}$$

About tangent. By ||al axes theorem

$I_{\text{tangent}} = I_d + Ma^2$

$= \frac{1}{2}Ma^2 + Ma^2 = \frac{3}{2}Ma^2$

(iii) About an axis passing through centre & ⊥ar to plane of ring.

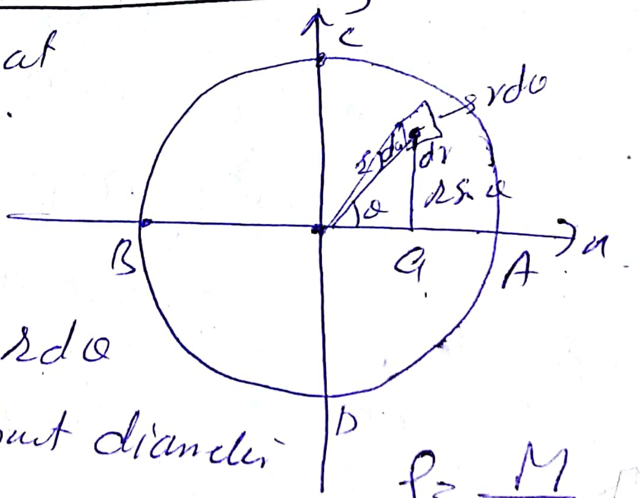
$$I_{oz} = I_x + I_y$$
$$= \frac{1}{2}Ma^2 + \frac{1}{2}Ma^2 = Ma^2$$

(iv) About an axis ⊥ar at any point of circumference $= I_{oz} + Ma^2$

$$= Ma^2 + Ma^2 = 2Ma^2.$$

Uniform circular Disc

Consider a patch at point (r, θ) in plane.
Let t be thickness



Volume of patch
$= dV = t \times dr \times r d\theta$

(i) Moment of inertia about diameter

$$\rho = \frac{M}{\text{Volume}} = \frac{M}{\pi a^2}$$

$$I_{AB} = I_{xx} = \int_{Disc} r^2 \sin^2\theta \, dm.$$

$$= \int_0^{2\pi}\int_0^{r} r^2 \sin^2\theta \cdot \rho \cdot t \cdot dr \, r d\theta$$

$$= \rho t \int_0^{2\pi}\int_0^{r} r^3 \, dr \, d\theta \cdot \sin^2\theta$$

$$= \rho t \left[\frac{r^4}{4}\right]_0^a \int_0^{2\pi} \sin^2\theta \, d\theta$$

$$= \frac{1}{4}\rho t \cdot a^4 \cdot 2 \cdot 2 \int_0^{\pi/2} \sin^2\theta \, d\theta = \rho t a^4 \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\rho = \frac{M}{\pi a^2 t}$$

$$I_d = \frac{M}{\pi a^2 t} \cdot t \cdot a^4 \cdot \frac{\pi}{4} = \boxed{\frac{1}{4} M a^2}$$

(2) M·I about central axis ⊥ar to Disc

$$I_{oz} = \int_{Disc} r^2 dm = \int_0^{2\pi} \int_0^{a} \rho r^2 t \, r \, dr \, d\theta$$

$$= \rho t \int_0^{2\pi} \int_0^{a} r^3 \, dr \, d\theta$$

$$= \rho t \left[\frac{r^4}{4}\right]_0^a \int_0^{2\pi} d\theta$$

$$= \frac{M}{\pi a^2 t} \cdot t \left[\frac{a^4}{4}\right] \cdot 2\pi$$

$$I_{oz} = \boxed{\frac{1}{2} M a^2}$$

(3) M·I about tangent.

$$I_t = I_d + M a^2$$

$$= \frac{1}{4} M a^2 + M a^2 = \frac{5}{4} M a^2$$

(4) About an axis through circumference & ⊥ar to Disc $= I_{oz} + M a^2$

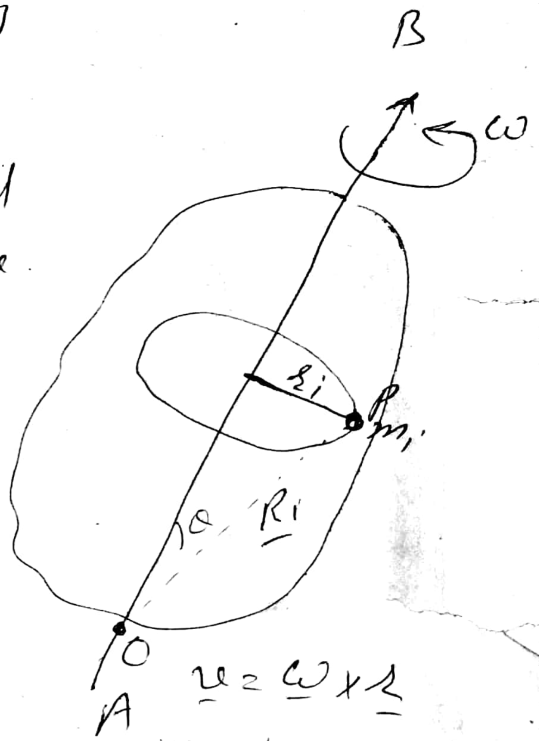
$$= \frac{1}{2} M a^2 + M a^2 = \frac{3}{2} M a^2$$

Motion of Rigid body

(1) K·E about a fixed Axis

If a rigid body rotates about a fixed axis with angular velocity ω, then K-E of rotation is given by

$$T = \frac{1}{2} I \omega^2 .$$



$\underline{v} = \underline{\omega} \times \underline{r}$

$|\underline{v}| = |\underline{\omega}||\underline{R}_i| \sin\theta$

$= \omega r_i$

Proof. Let AB be fixed axis of rotation. A particle of mass m_i will rotate about the axis with angular speed ω. The particle will describe a circle with linear speed

$$v_i = \omega r_i$$

$$K \cdot E \text{ of rotation} = \frac{1}{2} m_i v_i^2$$

$$= \frac{1}{2} m_i \omega^2 r_i^2$$

Total K.E of rotation is

$$T = \frac{1}{2} \omega^2 \sum m_i r_i^2$$

$$= \frac{1}{2} I \omega^2$$

where $I = \sum m_i r_i^2$ M.I about AB

(2) Angular Momentum about fixed Axis

The angular momentum of particle m_i about AB is

$m|R_i \times v_i|$

$m_i R_i v_i$

$m v r$.

$$L_i = m_i r_i v_i$$

$$= m_i r_i \omega r_i$$

$$= m_i r_i^2 \omega$$

Total angular momentum.

$$L = \sum m_i r_i^2 \omega$$

$$\boxed{L = I\omega}$$

Note $N = \tau = \frac{dL}{dt} = \frac{d}{dt}(I\omega)$

$$= I\frac{d\omega}{dt} = I\alpha$$

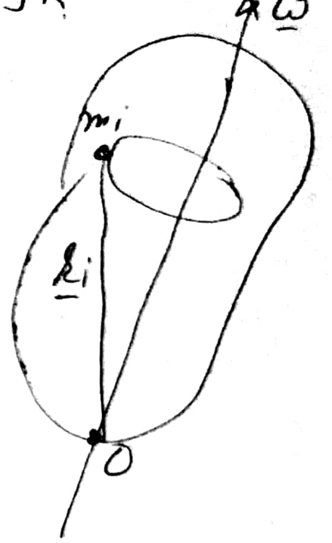
where α is angular acc.

General Motion of Rigid Body in space.

(1) The no of degrees of freedom for general motion in space is i.e 6 coordinates are required. We usually choose 3 of these to be coordinates of a point in the body [usually c.m] and remaining 3 to be angles [e.g Euler's angle] which describe rotation of rigid b about a point

3. Velocity and angular Momentum of Rigid body with one point fixed. (instantaneous axis of rotation)

Suppose point O of the rigid body R is fixed. Then at a given instant of time the body will be rotating with angular velocity $\underline{\omega}$ about the instantaneous axis through O. A particle m_i of body with p.v $\underline{r}_i$ w.r.t. O will have instantaneous velocity



$$\underline{v}_i = \dot{\underline{r}}_i = \underline{\omega} \times \underline{r}_i$$

The angular momentum of rigid body with one point fixed about instantaneous axis through the fixed point is

$$\underline{L} = \sum \underline{r}_i \times m_i \underline{v}_i$$

$$= \sum m_i (\underline{r}_i \times \underline{v}_i)$$

$$= \sum m_i [\underline{r}_i \times (\underline{\omega} \times \underline{r}_i)] \rightarrow (1)$$

Let $\underline{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$ with fixed co-ordinate syst xyz with origin at O

$\underline{L} = L_x \hat{i} + L_y \hat{j} + L_z \hat{k}$

$\underline{r}_i = x_i \hat{i} + y_i \hat{j} + z_i \hat{k}$

$\underline{r_i} \times (\underline{\omega} \times \underline{r_i})$ 4

$\Rightarrow \underline{\omega}(\underline{r_i} \cdot \underline{r_i}) - \underline{r_i}(\underline{\omega} \cdot \underline{r_i})$

$= \underline{\omega}(x_i^2 + y_i^2 + z_i^2) - \underline{r_i}(\omega_x x_i + \omega_y y_i + \omega_z z_i)$

$= (\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k})(x_i^2 + y_i^2 + z_i^2)$

$\quad - (x_i \hat{i} + y_i \hat{j} + z_i \hat{k})(\omega_x x_i + \omega_y y_i + \omega_z z_i)$

$= [\omega_x (y_i^2 + z_i^2) - \omega_y x_i y_i - \omega_z x_i z_i]\hat{i}$

$+ [\omega_y (x_i^2 + z_i^2) - \omega_x x_i y_i - \omega_z y_i z_i]\hat{j}$

$+ [\omega_z (x_i^2 + y_i^2) - \omega_x x_i z_i - \omega_y y_i z_i]\hat{k}$

putting in (1)

$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$

$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$

$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$

Proof $T = \frac{1}{2}\underline{\omega} \cdot \underline{L}$

If a rigid body with one point fixed rotates with angular velocity ω and has angular momentum $\underline{L}$, then prove that $T = \frac{1}{2}\omega \underline{L}$

$T = \frac{1}{2}\sum m_i v_i^2 = \frac{1}{2}\sum m_i(\dot{\underline{r}}_i \cdot \dot{\underline{r}}_i)$

$= \frac{1}{2}\sum m_i[(\underline{\omega} \times \underline{r_i}) \cdot (\underline{\omega} \times \underline{r_i})]$

using $(\underline{A}\times\underline{B})\cdot(\underline{C}\times\underline{D}) = \begin{vmatrix} \underline{A}\cdot\underline{C} & \underline{A}\cdot\underline{D} \\ \underline{B}\cdot\underline{C} & \underline{B}\cdot\underline{D} \end{vmatrix}$

$= (\underline{A}\cdot\underline{C})(\underline{B}\cdot\underline{D}) - (\underline{A}\cdot\underline{D})(\underline{B}\cdot\underline{C})$

$= \underline{A}\cdot[\underline{C}(\underline{B}\cdot\underline{D}) - \underline{D}(\underline{B}\cdot\underline{C})]$ $\quad \underline{B}\times(\underline{C}\times\underline{D})$

$= \underline{A}\cdot[\underline{B}\times(\underline{C}\times\underline{D})]$

$T = \frac{1}{2}\sum m_i\, \underline{\omega}\cdot[\underline{r}_i\times(\underline{\omega}_i\times\underline{r}_i)]$

$= \frac{1}{2}\underline{\omega}\cdot \sum m_i[\underline{r}_i\times(\underline{\omega}_i\times\underline{r}_i)]$

$= \frac{1}{2}\underline{\omega}\cdot\sum m_i(\underline{r}_i\times\underline{v}_i)$

$= \frac{1}{2}\underline{\omega}\cdot\underline{L}$

$= \frac{1}{2}[I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 + 2I_{xy}\omega_x\omega_y + 2I_{xz}\omega_x\omega_z + 2I_{yz}\omega_y\omega_z]$

M. I Matrix or Tensor

$I_{xy} = -\sum m_i x_i y_i = I_{yx}$

$I_{yz} = -\sum m_i y_i z_i = I_{zy}$

$I_{xz} = -\sum m_i x_i z_i = I_{zx}$

$\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$ is inertia tensor or matrix. Due to symmetry it is called symmetric matrix.

6

Principal Axes of Inertia

A set of mutually ⊥ar axes which are fixed in body and rotate with it and which are such that the products of inertia about them are zero are called principal axes of inertia or principal axes of body

Note An important property of a principal axis is (which can be taken as definition) is that if a rigid rotates about it the direction the angular momentum is same as that of angular velocity

$$L = I\omega$$

(1) Angular momentum about principal Axes

$$L_x = I_x\omega_x \qquad L_y = I_y\omega_y \qquad L_z = I_z\omega_z$$

(2) K.E about the principal axes is given by

$$T = \frac{1}{2}\left[I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2\right]$$

Ellipsoid of Inertia

$$\begin{bmatrix} A & D & E \\ D & B & C \end{bmatrix}$$

~~$I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 + 2I_{xy}$~~

$$Ax^2 + By^2 + Cz^2 + 2xyD + 2xzE + 2yzF = 1$$

$$\begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix}$$

where $D = -\sum m_i x_i y_i$

$F = -\sum m_i y_i z_i$

$E = -\sum x_i z_i m_i$

$A = \sum m_i(y_i^2 + z_i^2)$

$B = \sum m_i(x_i^2 + z_i^2)$

$C = I_{zz} = \sum m_i(x_i^2 + y_i^2)$

2

Euler's Equation of Motion

$$\tau_1 = N_1 = I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3$$
$$N_2 = I_2\dot{\omega}_2 - (I_3 - I_1)\,\omega_3\omega_1$$
$$N_3 = I_3\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2$$

Eulers Angles

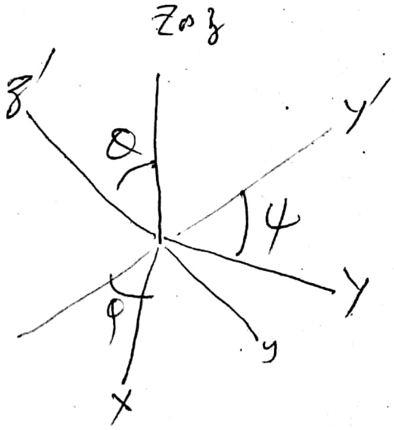
To describe the rotation of a rigid body about a point we use three angular co-ordinates φ, θ, ψ called Eulers angles

The space axes xyz are rotated into body axes $x'y'z'$ (usually principal) by three successive rotations

(1) Rotation about z-axis of x & y-axis through angle φ to get XYZ axis

(2) The XYZ system is rotated about X-axis through angle θ to get $X_1 Y_1 Z_1$

(3) finally rotate about z_1-axis through ψ.



Top – a rigid body capable of rotation about an axis is generally called top

<u>Symmetric top</u> If a rigid body is capable of rotation about a symmetry axis, then it is

Symmetrical top or if $I_1 = I_2 \neq I_3$, the body is symmetrical top

Asymmetrical Top

If the principal moments of inertia are all distinct i.e. if $I_1 \neq I_2 \neq I_3$ i.e body is not symmetric about any axis, then is called asymmetric.

Rotor If a body has $I_1 = 0, I_2 = I_3$ it is called rotor